

Random walks and the Hagedorn transition

Martin Kruczenski^{ab} and Albion Lawrence^a

^a*Martin Fisher School of Physics, Brandeis University
MS057, PO Box 549110, Waltham, MA 02454, U.S.A.*

^b*Dept. of Physics, Princeton University
Princeton, NJ 08544, U.S.A. (Present address)*

E-mail: martink@princeton.edu, albion@brandeis.edu

ABSTRACT: We study details of the approach to the Hagedorn temperature in string theory in various static spacetime backgrounds. We show that the partition function for a *single* string at finite temperature is the torus amplitude restricted to unit winding around Euclidean time. We use worldsheet path integral to derive the statement that the sum over random walks of the thermal scalar near the Hagedorn transition is precisely the image under a modular transformation of the sum over spatial configurations of a single highly excited string. We compute the radius of gyration of thermally excited strings in $AdS_D \times S^n$. We show that the winding mode indicates an instability despite the AdS curvature at large radius, and that the negative mass squared decreases with decreasing AdS radius, much like the type 0 tachyon. We add further arguments to statements by Barbón and Rabinovici, and by Adams *et. al.*, that the Euclidean AdS black hole can be thought of as a condensate of the thermal scalar. We use this to provide circumstantial evidence that the condensation of the thermal scalar decouples closed string modes.

KEYWORDS: Tachyon Condensation, AdS-CFT Correspondence, Black Holes in String Theory.

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1. Introduction

Polchinski [1] has shown that the free energy of a gas of noninteracting strings on a spatial manifold M at temperature $T = \beta^{-1}$ is the partition function of a single string on a torus, in the Euclidean background $M \times S^1_\beta$. Here the circle S^1_β is the Euclidean time direction and has radius β . As the temperature increases from below, there is a scalar string state known as the “thermal scalar” with unit winding around S^1_β which becomes a spacetime tachyon [2–4] at the “Hagedorn temperature” $T_H \sim m_s = \ell_s^{-1}$. (Here ℓ_s is the string scale and m_s^2 is the string tension). Near the Hagedorn temperature, this scalar dominates the partition function [4–6]. If there are noncompact spatial dimensions, then the divergence of the partition function near a phase transition arises from the infrared divergence of the one-loop contribution of the tachyon. Even if all dimensions are compactified, the one-loop partition function for the string gas diverges as $\ln(T_H - T)$ due to the appearance of the tachyon.

On the other hand, in the presence of noncompact spatial dimensions, an ensemble of strings at fixed energy is dominated by configurations with a *single* long string and a gas of small, light strings [7–9]. The radius of gyration of a long string with energy E and

tension ℓ_s^{-2} is $\langle r^2 \rangle \sim \ell_s^3 E$, indicating that the statistics of these strings are those of random walks of length $L = \ell_s^2 E$.

The thermal scalar seems like a formal device, and it would be nice to better understand the relation between the canonical and microcanonical descriptions of a gas of strings. One hint appears in [6, 10]: if one computes the one-loop partition function of the thermal scalar near the phase transition, it can be rewritten as the partition function of a single string with density of states

$$\rho(E) = \frac{e^{\beta_H E}}{E^{1+\alpha}}, \tag{1.1}$$

where α depends on the details of the worldsheet conformal field theory, such as the number of noncompact dimensions.¹ Furthermore, the correlation function of thermal scalars can be computed via a sum over random walks.

Atick and Witten [4] argued, by analogy with the deconfinement transition in QCD, that this transition should give some insight into the fundamental degrees of freedom of string theory. Since equilibrium thermodynamics in flat space fails in the presence of gravity, it is hard to know how far one can go with the perturbative flat space calculations. Anti-de Sitter (AdS) space, however, provides a convenient “box” for gravity, and the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [11] gives a definite prescription for studying string theory at finite temperature in such backgrounds. In light of this correspondence, one finds that there is indeed a relation between the Hagedorn transition and deconfinement [12–14, 16–19, 15, 20], which we will explore further below.

The goal of our paper is twofold. First, we will make the relation between the thermal scalar and random walks explicit, expanding on [6, 10], and apply our lessons to strings in AdS backgrounds. Next, we will tie together some of our own calculations for string theory in finite-temperature AdS backgrounds with previous work by other authors to flesh a picture of the Hagedorn transition in AdS backgrounds.

More precisely, in section 2 we show that the partition function for a single string at finite temperature can be written as the worldsheet torus amplitude restricted to unit winding around the torus. If there are “noncompact” directions in the target space — directions in which the spectrum of the momentum operator is continuous — then the single-string dominance of the microcanonical ensemble and the leading infrared singularity of the thermal scalar contribution near the Hagedorn temperature are the same effect, related by a worldsheet modular transformation. In completely compactified spacetimes, or spacetimes like $AdS_D \times S^n$ for which the momentum operator is discrete, single string dominance fails to hold in the microcanonical ensemble, while in the canonical ensemble the large-volume divergence of the thermal scalar is stripped away leaving the divergence due to the existence of an unstable mode. In section 3 we explicitly derive, in a wide class of static spacetimes, the statement that the sum over the spatial configurations of highly energetic strings is precisely a sum over random walks. We compute the size of highly excited strings in D -dimensional anti-de Sitter space (AdS_D) times an n -sphere S^n , as a

¹For the purposes of this paper, “noncompact dimensions” is a code for the presence of a continuous spectrum of conformal dimensions on the worldsheet, or of a continuous spectrum for the momentum operator. From this perspective, for example, anti-de Sitter space should be considered “compact”.

function of temperature; and we show that the tachyon indicates a genuine instability in AdS spacetimes. In section 4 we extend arguments of [21, 22, 10, 23] that the endpoint of the condensation of the tachyon is the AdS black hole. We then provide some circumstantial evidence that the closed string modes decouple in the presence of the tachyon in much the same way that open string excitations decouple when the open string tachyon condenses [24–35]. In section 5 we give a brief conclusion.

We should note that many of the basic statements in section 2, 3 are contained implicitly or explicitly in [6, 10]; while many of the statements in section 4.2 are contained in recent work such as [21, 22, 10, 23, 36, 37], and are surely known to others. However, we feel that our explicit derivation, the expanded discussion, and consolidation are worth putting into print. Other recent work on the relation between excited strings and random walks is [38].

2. The partition function for single strings

Polchinski showed that the partition function for a gas of strings in the canonical ensemble at temperature $T = \beta^{-1}$ can be computed as follows. Let the worldsheet conformal field theory consist of a $c = 25$ (for the bosonic string) or $\hat{c} = 9$ (for the superstring) factor for the spatial directions of the target space, times a $c = 1$ or $\hat{c} = 1$ CFT for the target space circle S^1_β with radius β . The free energy $\beta F(\beta)$ is the vacuum torus amplitude for the string in this Euclidean background [1]. One indication of this is that the amplitude can be written as the sum over closed string modes of the free energy for a gas of particles in each mode.

We will argue that the partition function for a *single* string at temperature β is the vacuum amplitude in Euclidean space with periodic Euclidean time, restricted to unit winding around S^1_β .

2.1 Partition function for a single string

Consider a scalar particle, or a particle in a fixed eigenstate of spin angular momentum, with mass m in flat space. The Euclidean spacetime is $M_{d+1} = S^1_\beta \times \mathbf{R}^d$. The partition function for one particle (as opposed to a gas of particles) with mass m in d spacetime dimensions is:

$$Z = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{-\beta\omega_k} \quad \omega_k^2 = \vec{k}^2 + m^2 \tag{2.1}$$

This can be written as a space-time integral:²

$$\begin{aligned} Z &= \int \frac{d^d k}{(2\pi)^d} \frac{2ik^0 e^{i\beta k^0}}{k^2 + m^2} ; \quad (k^2 = (k^0)^2 + \vec{k}^2 + m^2) \\ &= \int \frac{d^d k}{(2\pi)^d} \int_0^\infty ds (2ik^0) e^{i\beta k^0 - s((k^0)^2 + \vec{k}^2 + m^2)} \end{aligned}$$

²This relation can be derived in various ways. An alternative is to use the identity $e^{-\beta\sqrt{m^2+k^2}} = \frac{\beta}{\sqrt{2\pi}} \int_0^\infty \frac{ds}{s^{3/2}} e^{-\frac{1}{2}(m^2+k^2)s} e^{-\frac{\beta^2}{2s}}$.

$$\begin{aligned}
 &= 2\partial\beta \int_0^\infty \frac{ds}{(2\pi s)^{d/2}} e^{-\frac{\beta^2}{4s} - m^2 s} \\
 &= -\beta \int_0^\infty \frac{ds}{s(2\pi s)^{d/2}} e^{-\frac{\beta^2}{4s} - m^2 s}
 \end{aligned} \tag{2.2}$$

For a single string, one can sum this answer over all possible string states. The only difference between this calculation and that of [1] is that we are looking at worldlines that wrap *once* around the Euclidean time direction. Therefore, we can write the partition function for a single string at spacetime temperature $T = \beta^{-1}$ as:

$$Z_{s.s.} = \int_0^\infty \frac{d\tau}{4\pi\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 I_{(0,1)}(\tau) \tag{2.3}$$

where $I_{(m,n)}(\tau)$ is the one-loop partition function for the conformal field theory on the torus with modular parameter $\tau = \tau_1 + i\tau_2$, coordinates $\sigma_1 \in [0, 1], \sigma_2 \in [0, 1]$, worldsheet metric $ds^2 = |d\sigma_1 + \tau d\sigma_2|^2$, and boundary conditions

$$\begin{aligned}
 X^\mu(\sigma_1 + 1, \sigma_2) &= X^\mu(\sigma_1, \sigma_2) + m\beta\delta^{\mu,0} \\
 X^\mu(\sigma_1, \sigma_2 + 1) &= X^\mu(\sigma_1, \sigma_2) + n\beta\delta^{\mu,0} .
 \end{aligned} \tag{2.4}$$

Note that the integral is not over the fundamental domain of τ but rather along the entire strip $\tau_2 \geq 0, -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}$. This is the fundamental domain of the subgroup $\tau \rightarrow \tau + 1$ of the full modular group. Under this transformation (c.f. [39]) the winding numbers (m, n) are mapped to $(m, n - m)$. Therefore the $(0, 1)$ sector is invariant under this shift and one should restrict the modular integral over $(0, 1)$ to an integral over this strip.

One may further include the effects of the modular transformation $\tau \rightarrow 1/\tau$. Under this transformation, however, (m, n) is mapped to $(n, -m)$; in particular $(0, 1)$ is mapped to $(1, 0)$. The $(0, 1)$ sector is not invariant under the full modular group $SL(2, \mathbb{Z})$. Its image is the set of *coprime* integers (m, n) . One may therefore rewrite the above partition function as an integral over the fundamental domain $\mathcal{F} = \{\tau \mid |\tau_1| \leq \frac{1}{2}, |\tau|^2 \geq 1\}$, and a sum over $I_{m,n}$ with (m, n) coprime:

$$Z_{s.s.} = -\beta \int_{\mathcal{F}} \frac{d^2\tau}{4\pi\tau_2^2} \sum_{(m,n) \text{ coprime}} I_{(m,n)}(\tau) . \tag{2.5}$$

2.2 Single string dominance and the Hagedorn transition

If we sum (2.5) over all $(m, n) \in \mathbb{Z}^2$, the result is $-\beta$ times the free energy for a *gas* of strings at temperature β^{-1} [1]. If Z_k is the partition function for k strings, and we fix our normalization so that $Z_0 = 1$, then the free energy can be written as

$$\begin{aligned}
 F &= -\frac{1}{\beta} \ln(1 + Z_1 + Z_2 + \dots) \\
 &= -\frac{1}{\beta} \left(Z_1 + \left(-\frac{1}{2}Z_1^2 + Z_2 \right) + \dots \right) \\
 &= -\frac{1}{\beta} (Z_{[0,1]} + Z_{[0,2]} + \dots) .
 \end{aligned} \tag{2.6}$$

Here $Z_{[0,k]}$ comes from replacing $I_{(0,1)}$ in (2.3) with $I_{(0,k)}$, and $Z_{[0,1]} = Z_{s.s.}$. We will demonstrate in appendix A that $Z_2 - \frac{1}{2}Z_1^2 = Z_{[0,2]}$. $Z_{[0,k]}$ looks effectively like the partition function for a single string at temperature $1/(k\beta)$. Therefore, $Z_{[0,1]} = Z_{s.s.}$ dominates the free energy for a gas of strings at the Hagedorn temperature. We should note that the partition function $Z = \sum_k Z_k$ generally contains divergences from each sector Z_k , although when there are noncompact spatial dimensions (e.g. the spectrum of conformal dimensions is continuous) the divergences will be subleading for $k > 1$: it is well known that single strings dominate the partition function for very high energies [7–9], and it is the high energy states which dominate near the Hagedorn transition.

Strings in the sector $(m, n) = (1, 0)$ become tachyonic at the ‘‘Hagedorn temperature’’ $(\beta_H)^{-1} = (2\sqrt{2}\pi\ell_s)^{-1}$. If the number of noncompact spatial dimensions is nonzero, this leads to an infrared divergence in the partition function characteristic of a massless scalar field, characteristic of a phase transition [4]. This divergence is related under $\tau \rightarrow \frac{-1}{\tau}$ to the ultraviolet divergence coming from the $(m, n) = (0, 1)$ sector; this divergence arises from the large UV density of states of a single string. In other words, the divergence in Z_1 is mapped by a modular transformation to the infrared divergence in the free energy for the thermal scalar. We will explore this map in detail in section 3.

Furthermore, the large-volume infrared divergence for the thermal tachyon is cut off when all directions are compactified. It is known that in this case, a single long string no longer dominates the ensemble at large energies [9, 10]. One is left with a divergence due entirely to the negative eigenvalue of the tachyon kinetic term.

In short, in the presence of noncompact dimensions, the infrared divergence in F due to the thermal scalar at the Hagedorn temperature is a re-encoding of single-string dominance, via a worldsheet modular transform.

3. The random walk picture for excited strings

3.1 Review of the random walk picture of the thermal scalar

Studies of a gas of strings in the microcanonical ensemble [7–9] have shown that the single string which dominates at large energies has a radius of gyration r_g characteristic of a random walk, $r_g^2 = \langle \delta x^2 \rangle \sim \ell_s L = \ell_s^3 E$, where L is the length of the string, E the energy, and $m_s^2 = \ell_s^{-2}$ the string tension. Horowitz and Polchinski [6] have also argued this from the canonical ensemble. The essence of their argument can be summarized as follows. The quadratic part of the thermal scalar Lagrangian is $-\frac{1}{2} \int d^d x \phi D \phi$, with $D = \nabla^2 + m_\beta^2$ and $m_\beta^2 = \frac{\beta_H^2 - \beta^2}{4\pi(\alpha')^2}$ [6]. The equation for the heat kernel of D :

$$(\nabla^2 + m_\beta^2)K = \ell_s \partial_T K; \quad K(\vec{x}, T = 0) = \delta^{(d)}(\vec{x}) \tag{3.1}$$

can be solved by writing $K = e^{m^2 \ell_s T} P$. Now,

$$P = \frac{e^{-\frac{|x|^2}{2\ell_s T}}}{(\ell_s T)^{d/2}} \tag{3.2}$$

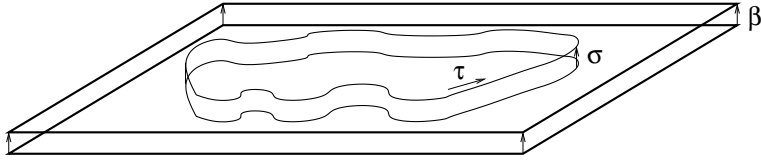


Figure 1: The path integral for a single string close to the Hagedorn temperature can be written as sums over random walks of the thermal winding mode, or as random spatial configurations of the string which wraps once around the thermal time direction.

solves the diffusion equation in d dimensions, which is also the equation for the probability distribution of the position of a random walk of length T which begins at the origin. Near $m_\beta^2 = 0$, we can write

$$m_\beta^2 = \frac{\beta_H^2 - \beta^2}{4\pi^2(\alpha')^2} \sim \frac{\beta_H(\beta_H - \beta)}{2\pi^2(\alpha')^2} \quad (3.3)$$

Therefore, we find that

$$K = e^{\frac{\beta_H^2}{2\pi(\alpha')^2}\ell_s T} P(\vec{x}, T) e^{-\frac{\beta_H \ell_s T}{2\pi(\alpha')^2}\beta} \quad (3.4)$$

Here $\beta_H = \alpha'$, the first factor $e^{\beta_H^2 \ell_s T / 2\pi(\alpha')^2}$ on the right hand side is the total number of random walks of length T , and the third factor $e^{-\beta_H \beta \ell_s T / 2\pi(\alpha')^2}$ is a Boltzmann suppression factor.

The free energy for the thermal scalar is

$$F = -\beta \ln Z = -\beta \int_0^\infty \frac{dT}{T} K(0, T) \quad (3.5)$$

Here $e^{\beta_H^2 \ell_s T / 2\pi(\alpha')^2} P(0, T) / T$ is the number of closed paths of length T : the extra factor of $1/T$ compensates for the fact that one may choose any point on the loop as the starting place for the walk. The result is that the free energy for the thermal scalar is precisely the partition function for a single static string with tension $1/2\pi(\alpha')$.

We would like to relate this result to the sum over massive string states in the following way. As discussed in section 2, the worldsheet path integral on the torus with winding number $(1, 0)$ contains the path integral over the thermal scalar. Close to the Hagedorn temperature, the path integral is dominated by large values of τ_2 . We identify this with the Schwinger parameter T in the discussion above. However, as we noted in section 2, the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ takes this path integral to one over a string with winding number $(0, 1)$. Large τ_2 is mapped to small τ_2 . The sum over paths of the thermal scalar are mapped to sums over spatial configurations of the string, which wraps the thermal circle once as in figure 1. The random walks summed over in the thermal scalar path integral should be precisely the spatial configurations of the string in the first-quantized picture.

We would like to understand the degree to which this picture is precise. In the string theory path integral, one can write worldsheet coordinates t, σ such that the winding of the string is in the t direction. A simple sum over random walks would be a sum over string worldsheets for which the target space coordinates of the string depend only on σ : however, in general the coordinates can depend both on the spatial coordinate σ on the worldsheet,

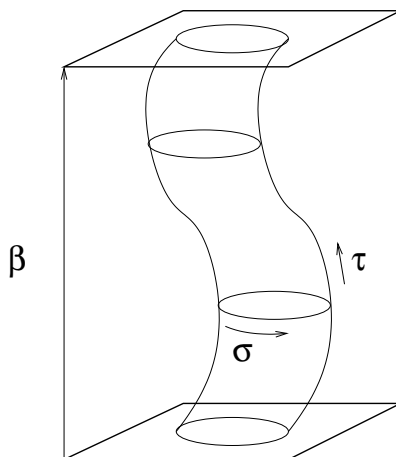


Figure 2: Periodic worldsheet contributing to the finite temperature path integral (the upper and lower surfaces are identified). At temperatures away from the Hagedorn temperature, the string oscillations will depend on Euclidean time.

and on the time direction t , as in figure 2. Nonetheless, for generic highly massive strings we will directly derive this picture in a way that will generalize to strings in a large class of static curved spacetimes, such as anti-de Sitter space, so we can compute the radius of gyration of a string as a function of temperature in these spacetimes as well. Our general results will be essentially those stated in [6], though we hope that our explicit derivation and discussion of its validity will be of use. We will go on to derive the radius of gyration of strings in anti-de Sitter space at finite temperature.

3.2 Path integral for highly excited strings in curved spacetimes

We study string theory on the target space $S^1_\beta \times M_9$, parameterized by coordinates $X^0, \dots, d-1$, where $X^0 \equiv X^0 + 2\pi\beta$. We will take target space metrics of the form:

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = G_{00}(X^i)(dX^0)^2 + G_{ij}(X^i) dX^i dX^j \tag{3.6}$$

In particular we assume that translations in X_0 are isometries of the metric. We are interested in configurations such that $X^\mu(\sigma^1+1, \sigma^2) = X^\mu$, $X^0(\sigma^1, \sigma^2+1) = \beta + X^0(\sigma^1, \sigma^2)$, and $X^i(\sigma^1, \sigma^2+1) = X^i(\sigma^1, \sigma^2)$. The Polyakov action for strings on the torus in conformal gauge is:

$$S = m_s^2 \int d^2\sigma \left[\frac{|\tau|^2}{\tau_2} G_{\mu\nu} \partial_1 X^\mu \partial_1 X^\nu - 2 \frac{\tau_1}{\tau_2} G_{\mu\nu} \partial_1 X^\mu \partial_2 X^\nu + \frac{1}{\tau_2} G_{\mu\nu} \partial_2 X^\mu \partial_2 X^\nu \right]$$

We wish to study the full partition function figure

$$Z = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2} \Delta_{FP} \int \mathcal{D}X \prod_{\sigma^1, \sigma^2} \sqrt{G(X(\sigma^\alpha))} e^{-S[X]} \tag{3.7}$$

in the small τ_2 region of the integrand, which dominates for highly excited string states. Note that we have kept the \sqrt{G} measure factor explicit, so that the path integral measure

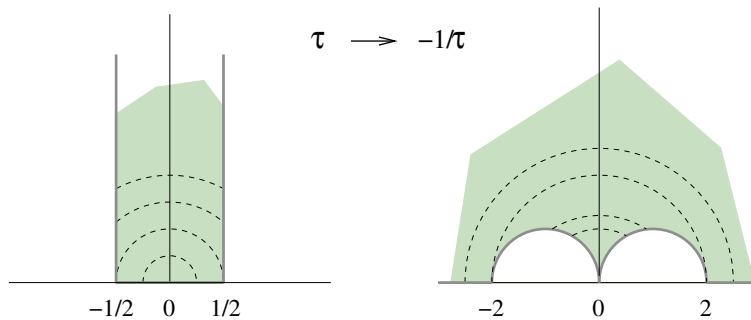


Figure 3: The domain $|\Im m\tau| \leq \frac{1}{2}$ transforms under $\tau \rightarrow -1/\tau$ as shown. A simple way to understand the mapping is that circles around the origin map to circles around the origin but with inverse radius (see the dashed circles in the figure).

is invariant under reparameterizations of the target space. The Faddeev-Popov determinant Δ_{FP} is given by

$$\Delta_{FP} = \frac{1}{\tau_2} \det' \Delta \quad (3.8)$$

with Δ the worldsheet Laplacian. The prime in the determinant indicates that we have removed the zero mode.

We will make a change of variables $\tau \rightarrow -\frac{1}{\tau}$ in the integrand, which is of course just a modular transformation. Note that the τ measure $\frac{d^2\tau}{\tau_2^2}$, the Faddeev-Popov determinant Δ_{FP} , and the path integral over X are separately modular invariant. The integration domain is mapped to that shown in figure 3. The boundary conditions on X^μ become: $X^\mu(\sigma^1, \sigma^2 + 1) = X^\mu$, $X^0(\sigma^1 + 1, \sigma^2) = \beta + X^0(\sigma^1, \sigma^2)$, and $X^i(\sigma^1 + 1, \sigma^2) = X^i(\sigma^1, \sigma^2)$, and we will set $X^0 = \beta\sigma^1 + \delta X^0$, where δX^0 is periodic on the torus. The resulting action on the torus is:

$$\begin{aligned} S = m_s^2 \int d^2\sigma \left[\frac{|\tau|^2}{\tau_2} (G_{00} (\beta^2 + 2\beta\partial_1 X^0 + (\partial_1 X^0)^2) + G_{ij} \partial_1 X^i \partial_1 X^j) \right. \\ \left. - 2\frac{\tau_1}{\tau_2} (G_{00} \beta \partial_2 \delta X^0 + G_{00} \partial_1 \delta X^0 \partial_2 \delta X^0 + G_{ij} \partial_1 X^i \partial_2 X^j) \right. \\ \left. + \frac{1}{\tau_2} (G_{00} (\partial_2 \delta X^0)^2 + G_{ij} \partial_2 X^i \partial_2 X^j) \right] \quad (3.9) \end{aligned}$$

Having made a the transformation $\tau \rightarrow -1/\tau$, we are interested in the limit of *large* τ_2 . In that limit from eq. (3.9) we see that configurations such that $\partial_1 \delta X^0 \neq 0$ and/or $\partial_1 X^i \neq 0$ are highly suppressed in the path integral. However, one cannot completely ignore the effects of these terms: in the limit of large τ_2 , the oscillator modes contribute to the vacuum energy and thus lead to a contribution $e^{c\pi/6}$ to the path integral, where c is the number of oscillators. We can combine this factor with the large- τ limit of Δ_{FP} :

$$\frac{1}{\tau_2} \det' \Delta \stackrel{\tau_2 \rightarrow \infty}{\sim} \tau_2 e^{-\frac{\pi}{3}\tau_2} \quad (3.10)$$

For the case of the superstring, the superconformal ghosts and fermion superpartners will lead to similar terms in the path integral. Their effect is to add a term of the form

$\delta S = -\int_0^1 d\sigma^2 m_s^2 \beta_H^2 \tau_2$ to the action. If we include this term and set all the ∂_1 derivatives to zero, we can write the resulting action as:

$$S = m_s^2 \int d\sigma_2 \left[\frac{1}{\tau_2} (G_{ij} \partial_2 X^i \partial_2 X^j) + \tau_2 (G_{00} \beta^2 - \beta_H^2) + \frac{\beta^2 G_{00}}{\tau_2} \left(\tau_1 - \frac{\partial_2 X^0}{\beta} \right)^2 \right] \quad (3.11)$$

The path integral over X^0 is Gaussian and can be evaluated exactly. The equation of motion for X^0 is

$$\partial_2 \left[G_{00} \left(\tau_1 - \frac{\partial_2 X^0}{\beta} \right) \right] = 0 \quad (3.12)$$

with solution

$$\partial_2 X^0 = \beta \tau_1 - \frac{C}{G_{00}} \quad (3.13)$$

where C is a constant of integration which can be found using the condition that X^0 is periodic in σ_2 :

$$0 = \int_0^1 d\sigma_2 \partial_2 X^0 \quad \Rightarrow \quad C = \beta \tau_1 \left(\int_0^1 \frac{d\sigma_2}{G_{00}} \right)^{-1} \equiv \beta \tau_1 \langle G_{00}^{-1} \rangle^{-1}, \quad (3.14)$$

where we define $\langle F \rangle = \int_0^1 d\sigma_2 F$. After we expand around this classical solution, the path integral over X_0 gives

$$\begin{aligned} \int \mathcal{D}X^0(\sigma_2) e^{-\frac{m_s^2}{\tau_2} \int d\sigma_2 G_{00} (\beta \tau_1 - \partial_2 X^0)^2} &= e^{-\frac{m_s^2 \beta^2 \tau_1^2}{\tau_2} \langle G_{00}^{-1} \rangle^{-1}} \int \mathcal{D}X^0(\sigma_2) e^{-\frac{m_s^2}{\tau_2} \int d\sigma_2 G_{00} (\partial_2 X^0)^2} \\ &= \mathcal{N}_0 \tau_2^{-\frac{1}{2}} \langle G_{00}^{-1} \rangle^{-\frac{1}{2}} e^{-\frac{m_s^2 \beta^2 \tau_1^2}{\tau_2} \langle G_{00}^{-1} \rangle^{-1}} \end{aligned} \quad (3.15)$$

where \mathcal{N}_0 is an infinite normalization factor that we are going to drop. (This integral can be done by discretizing σ_2 and taking the step size to zero at the end). We must, however, keep the $\sqrt{G_{00}}$ contribution to the \sqrt{G} part of the path integral measure. Since we are studying metrics of the form (3.6), we can factor it out of the integral over X^0 .

Replacing the result (3.15) in eq. (3.7) and performing the τ_1 integration we obtain

$$Z \simeq \frac{1}{\beta} \int_0^\infty \frac{d\tau_2}{\tau_2} \int_{X(0)=X(1)} \mathcal{D}X(\sigma_2) \prod_{\sigma_2} \sqrt{G_{00} \det G_{ij}(X(\sigma_2))} \times \quad (3.16)$$

$$\times e^{-m_s^2 \int_0^1 d\sigma_2 \left[\frac{1}{2\tau_2} (G_{ij} \partial_2 X^i \partial_2 X^j) + \tau_2 (\beta^2 G_{00}(X) - \beta_H^2) \right]}. \quad (3.17)$$

By writing $\sigma_2 = \tau_2 t$, we can write this as:

$$Z \simeq \frac{1}{\beta} \int_0^\infty \frac{d\tau_2}{\tau_2} \int_{X(\tau_2)=X(0)} \mathcal{D}X \sqrt{G_{00} \det G_{ij}} e^{-\int_0^{\tau_2} dt L(X)} \quad (3.18)$$

where

$$L = \frac{m_s^2}{2} G_{ij} \partial_t X^i \partial_t X^j + m_s^2 \beta^2 G_{00} - \beta_H^2. \quad (3.19)$$

This is clearly an integral over random walks of the thermal scalar. The paths are images under the modular transformation $\tau \rightarrow 1/\tau$ of configurations of strings at temperature β .

The path integral over X_i can in principle be computed in the following way. The integral over $\mathcal{D}X$ in (3.18) is the heat kernel for the thermal scalar:

$$K(\vec{X}_t, t; \vec{X}_0, 0) = \int_{X(0)=X_0, X(t)=X_t} \mathcal{D}X \sqrt{G_{00} G_{ij}} e^{-\int_0^t dt' L} \quad (3.20)$$

and solves the diffusion equation:

$$-\frac{1}{\sqrt{G_{00} \det G_{ij}}} \partial_i \sqrt{G_{00} \det G_{ij}} G^{ij} \partial_j K + (\beta^2 G_{00} - \beta_H^2) K \equiv H K = -\partial_t K \quad (3.21)$$

with boundary conditions

$$\lim_{t \rightarrow 0} K(X_t, t; X_0, 0) = \delta(X_t = X_0) \quad (3.22)$$

Note the factors of G_{00} in the wave equation; these arise from the explicit factors in the measure. The thermal partition function for a single string can be written (assuming it is dominated by large τ_2) as:

$$Z \simeq \frac{1}{\beta} \int_0^\infty \frac{d\tau_2}{\tau_2} \int dX_0 K(X_0, \tau_2; X_0, 0) \quad (3.23)$$

We can write K as a sum over eigenmodes of H :

$$K = \sum_n A_n \psi_n(X_t, X_0) e^{-E_n t} \quad (3.24)$$

this sum could be discrete or continuous. If H has a discrete spectrum with approximate spacing δ between energy levels, then for $t \gg 1/\delta$, K can be well-approximated by the ground state wavefunction for the tachyon. This is not the case for the string in flat space — K then has a continuous spectrum, and the string spreads without bound as $\tau_2 \rightarrow \infty$, as described in section 3.1.

3.3 The radius of gyration for highly excited strings

Given the propagator K as defined in (3.20), we can compute the typical size of a string as a function of the temperature β . We take as this typical size

$$(\delta x)^2 = \frac{1}{\beta Z} \int_0^\infty \frac{d\tau_2}{\tau_2} \int_0^{\tau_2} \frac{dt}{\tau_2} \langle (X(t) - X(0))^2 \rangle(\tau_2) \quad (3.25)$$

where

$$\begin{aligned} \langle (X(t) - X(0))^2 \rangle &= \int d^d X_t \sqrt{G_{00} \det G_{ij}(X_t)} \int d^d X_0 \sqrt{G_{00} \det G_{ij}(X_0)} \\ &\quad \times K(X_0, \tau_2; X_t, t) (X_t - X_0)^2 K(X_t, t; X_0, 0) \end{aligned} \quad (3.26)$$

As with the propagator itself, if the spectrum of H is discrete with level spacing δ , and the integral in (3.25) is dominated by sufficiently large $\tau_2 \gg 1/\delta$, we can approximate K by

the ground state wavefunction $A_0\psi_0e^{-E_0t}$. To see this, let us approximate K by the first two terms in (3.24):

$$K(X_2, t_2; X_1, t_1) = A_0\psi_0(X_2, X_1)e^{-E_0(t_2-t_1)} + A_1\psi_1(X_2, X_1)e^{-E_1(t_2-t_1)} + \dots \quad (3.27)$$

We can then perform the integral over t in (3.26):

$$\begin{aligned} \int_0^{\tau_2} \frac{dt}{\tau_2} K(X_0, \tau_2; X_t, t) K(X_t, t; X_0, 0) &= A_0^2\psi_0(X_0, X_t)\psi_0(X_t, X_0)e^{-E_0\tau_2} \\ &+ A_1^2\psi_1(X_0, X_t)\psi_1(X_t, X_0)e^{-E_1\tau_2} \\ &+ \frac{A_0A_1}{\tau_2(E_1 - E_0)} (e^{-E_0\tau_2} - e^{-E_1\tau_2}) \\ &\times (\psi_0(X_0, X_t)\psi_1(X_t, X_0) + \psi_1(X_0, X_t)\psi_0(X_t, X_0)) \end{aligned} \quad (3.28)$$

The first term clearly dominates when $\tau_2 \gg \frac{1}{E_1 - E_0}$. In this case, the radius of gyration $r_g = \sqrt{(\delta x)^2}$ can be well approximated by the width of the ground state wavefunction ψ_0 .

3.4 Excited strings in anti-de Sitter space

We would like to study the results of the previous section for the spacetime $AdS_D \times S^n$, where each factor has radius of curvature R , as we would find in Freund-Rubin-type compactifications of supergravity [40]. We will assume $R \gg \ell_s$, so that the worldsheet sigma model is weakly coupled. The metric for this spacetime is

$$ds^2 = - \left(1 + \frac{r^2}{R^2} \right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{R^2} \right)} + r^2 d\Omega_{D-2}^2 + R^2 d\Omega_n^2 \quad (3.29)$$

where $d\Omega_k^2$ is the metric for a k -sphere with unit radius. Note that these backgrounds in string theory typically have Ramond-Ramond tensor fields. While this implies a far more complicated worldsheet action than the one we are studying, we will ignore this effect and focus on the worldsheet action for the bosonic modes.

In this case, the propagator K satisfies the wave equation

$$\begin{aligned} \frac{-1}{r^{D-2}} \partial_r \left(r^{D-2} \left(1 + \frac{r^2}{R^2} \right) \right) \partial_r K - \frac{\nabla_{D-2}^2}{r^2} K \\ - \frac{\nabla_n^2}{R^2} K + \frac{\beta^2}{\ell_s^4} \left(1 + \frac{r^2}{R^2} \right) K - \frac{\beta_H^2}{\ell_s^4} K = -\partial_t K \end{aligned} \quad (3.30)$$

where ∇_k^2 is the Laplacian on the unit k -sphere. This equation is separable: if θ^a denote the angular coordinates in AdS_D and ψ^m the angular coordinates along S^n , then the solution to (3.30) satisfying the required boundary conditions is:

$$K(r, \theta, \psi) = K_D(r, \theta, t) K_n(\psi, t) \quad (3.31)$$

where

$$\begin{aligned} H_D K_D = \frac{-1}{r^{D-2}} \partial_r \left(r^{D-2} \left(1 + \frac{r^2}{R^2} \right) \right) \partial_r K_D - \frac{\nabla_{D-2}^2}{r^2} K_D \\ + \frac{\beta^2}{\ell_s^4} \left(1 + \frac{r^2}{R^2} \right) K_D - \frac{\beta_H^2}{\ell_s^4} K_D = -\partial_t K_D \end{aligned} \quad (3.32)$$

and

$$H_n K_n - \frac{\nabla_n^2}{R^2} K_n = -\partial_t K_n \tag{3.33}$$

Even without the additional tachyon mass term, the spectrum of the Laplacian in the above coordinates (covering all of AdS) is discrete. The tachyon wave equation, with the additional mass term will certainly have a discrete spectrum as well. Therefore, from the standpoint of string thermodynamics, this spacetime behaves as if the spatial directions are completely compactified — the power-law prefactor α in the density of states

$$\rho(E) = \frac{e^{\beta_H E}}{E^{1+\alpha}}$$

arises from target space directions which leads to a continuous spectrum of eigenvalues of the tachyon wave operator. In the case of AdS_D spacetimes, we expect that $\alpha = 0$ for energies high enough that the curvature of $AdS_D \times S^n$ begins to affect the string spectrum.

We can think of anti-de Sitter space as a hyperbolic space with a gravitational potential. Random walks in hyperbolic space tend to move towards the boundary at infinity [41], which is entropically favored due to the large volume there. However, we will see that the gravitational potential overwhelms this tendency, so that the radius of gyration for random walks asymptotes to a finite value.

For sufficiently small temperatures and energies, we expect the string to act like a string in flat space — its size $r_g = \sqrt{(\delta x)^2} \sim \sqrt{(\ell_s^3 M)}$ scales with the square root of the length $L \sim \ell_s^2 M$, where M is the mass of the string state. As the temperature and thus the average energy M increases, the string will spread. The integral over the modular parameter τ_2 has a saddle point at roughly $1/M$ (more precisely, under a modular transform $\tau = -1/\tau'$, the paths of the thermal scalar become string configurations, and $\tau' M$ where M is the mass of the excited string state). We will find that there are two characteristic temperatures at which the string starts to feel the curvature. At the lower temperature T_1 , the dominant values of τ_2 in (3.25) are long enough that K_D is well-approximated by its ground state wavefunction, and the string stops spreading in the AdS direction. At a higher temperature T_2 the string spreads over the entire S^n , and r_g can increase no further.

K_n satisfies the heat equation on S^n : solutions can be found, for example, by analytic continuation of the results in [42, 43] to Euclidean space. If s is the geodesic distance on S^n between X_t and X_0 , and R the radius, and $p = \cosh \frac{s}{R}$,

$$K(X_t, X_0; t) = \frac{1}{(2\pi R^2)^{\frac{n}{2}}} \left(\frac{d}{dp}\right)^{\frac{n}{2}-1} \sum_{k=0}^{\infty} P_k(p) \exp \left\{ -\frac{t}{R^2} \left(\left(k + \frac{1}{2}\right)^2 - \left(n - \frac{1}{2}\right)^2 \right) \right\} \tag{3.34}$$

where P_n are the Legendre polynomials. Note that the eigenvalues of ∂_t have spacing of order

$$\delta_n \sim 1/R^2,$$

as implied by (3.33).

K_D cannot be solved exactly. However, we can estimate the level spacing for the lowest eigenvalues of ∂_t , and the width of the corresponding wavefunctions, using scaling

arguments. Begin by defining dimensionless variables $x = \frac{r}{R}$. The eigenvalue equation for the left hand side of (3.32) is:

$$\frac{1}{R^2 x^{D-2}} \partial_r (x^{D-2} (1+x^2)) \partial_r K - \frac{\nabla_{D-2}^2}{R^2 x^2} K + \frac{\beta^2}{\ell_s^4} x^2 K + \frac{\delta\beta^2}{\ell_s^4} K = EK \quad (3.35)$$

where $\delta\beta^2 = \beta^2 - \beta_H^2$. Let us multiply the whole equation by R^2 . Defining dimensionless parameters $\mathcal{E} = R^2 E$,

$$\mu = \mathcal{E} - \frac{\delta\beta^2 R^2}{\ell_s^4}, \quad (3.36)$$

and

$$\alpha^2 = \frac{\beta^2 R^2}{\ell_s^4}$$

we find that the eigenvalue equation is completely controlled by α :

$$-\frac{1}{x^{D-2}} \partial_x (x^{D-2} (1+x^2)) \partial_x K - \frac{\nabla_{D-2}^2}{x^2} K + \alpha^2 x^2 K = \mu K \quad (3.37)$$

In the cases we are interested in, we expect α to be large. That is, we are interested in string theory on spaces for which $R/\ell_s \gg 1$. We may also tune β . However, we are interested in temperatures high enough so that the radius of gyration is significantly affected by the AdS background. We will find that such temperatures are (self-consistently) very close to $\beta_H^{-1} \sim m_s$.

In the limit of large α , the quadratic ‘‘potential’’ term in (3.37) should ensure that the eigenfunctions have a small width. If we define $x = \frac{y}{\sqrt{\alpha}}$, we can write the first term in (3.37) as

$$\frac{\alpha}{y^{D-2}} \partial_y \left(y^{D-2} \left(1 + \frac{y^2}{\alpha} \right) \right) \partial_y K \simeq \frac{\alpha}{y^{D-2}} \partial_y y^{D-2} \partial_y K \quad (3.38)$$

Therefore, we can rewrite (3.37) as

$$\mathcal{H}K = -\frac{1}{y^{D-2}} \partial_y y^{D-2} \partial_y K - \frac{\nabla_{D-2}^2}{y^2} K + y^2 K = \frac{\mu}{\alpha} K \quad (3.39)$$

In these units, we can expect that the low-lying eigenfunctions of \mathcal{H} have width $(\delta y)^2 \sim 1$ and eigenvalues $\mu/\alpha \sim 1$. Translated into our original dimensionful variables, we find that the width of the low-lying eigenfunctions of H_D is approximately:

$$(\delta r)^2 \sim \frac{R^2}{\alpha} = \frac{\ell_s^2 R}{\beta} \quad (3.40)$$

while the energy eigenvalues are of the order

$$E \sim \frac{\alpha}{R^2} + \frac{\delta\beta^2}{\ell_s^4} = \frac{\beta}{\ell_s^2 R} + \frac{\delta\beta^2}{\ell_s^4} \quad (3.41)$$

and the gap between energy levels can be expected to be of order

$$\delta_D \sim \frac{\alpha}{R^2} = \frac{\beta}{\ell_s^2 R}.$$

We can see from (3.40) a major difference between long strings in AdS space and flat space. In the latter case, the radius of gyration would diverge as $\beta \rightarrow \beta_H$ [6].

We expect that the picture of single strings at finite temperature is as follows. At low enough temperatures, the strings are small and do not feel the curvature of $AdS_D \times S^n$. The radius of gyration grows as the square root of their mass: $r_g^2 \sim \ell_s^3 M$. For highly massive strings in flat space, the integral over the modular parameter τ_2 has a saddle point at $\tau_2 \sim \frac{1}{\ell_s M}$. After the modular transform to the thermal scalar picture, and rescaling, this becomes $\tau_2 \sim M$. In the canonical ensemble for a single string in flat space, we can write (see for example [6]):

$$M = \frac{\ell_s}{\delta\beta^2} \tag{3.42}$$

Therefore the string becomes massive and large as β approaches $\beta_H \sim \ell_s$. When the temperature grows to temperature T_1 such that $M \sim \frac{R}{\ell_s \beta}$, the integral over τ_2 can be expected to have a saddle point at $\tau_2 \sim \ell_s^3 M = r_g^2 \sim \frac{\ell_s^2 R}{\beta}$. At this point, τ_2 is of the order of the level spacing δ_D for eigenvalues of the operator H_D . For higher temperatures, therefore, we expect that K is well-approximated by the ground state wavefunction and the string ceases to grow in the AdS_D direction. We should still be able to use (3.42) to estimate the temperature, as the flat space approximation is only just starting to fail: we find that $\delta\beta^2 \sim \frac{\ell_s^3}{R} \ll \ell_s^2$; the temperature is quite close to the Hagedorn temperature.

The string has stopped expanding in the radial direction of AdS_D because of the gravitational potential. However, the string can still expand in the S^n direction. We expect the expansion of the string in this direction to be essentially that of a string in \mathbb{R}^n , until the size of the string is of order the S^n radius R . We expect that at this point, the integral over τ_2 is dominated by values of order $M \ell_s^3 \sim r_g^2 \sim R^2$, and K_n is well-approximated by the constant mode on S^n .

At this point the string feels the curvature in all spatial directions, and for all intents and purposes sees the spacetime as being completely compactified. As discussed at the end of section 2, single string dominance will cease at this point, and multiple-string states will start to become equally likely.

One might have wondered whether the thermal scalar would lead to an instability above some temperature $T_H = \beta_H^{-1}$. AdS space allows for tachyons that do not indicate instabilities [44, 45]. Furthermore, in AdS space at finite temperature, the proper radius of the Euclidean time direction grows as one approaches the boundary of AdS. Thus the mass of the thermal scalar gets large and positive at large radius, after the fashion of the “localized” and “quasi-localized” tachyons discussed in [46, 23, 36], so that the tachyon is at best localized in the center of AdS space [21]. However, so long as a ground state wavefunction exists with the scaling properties we have described, we are guaranteed such an instability. For fixed μ, α , (3.37) is independent of the temperature, and a lowest value of μ/α will exist and be of order 1. However, the relationship between μ and E is temperature-dependent, leading to the statement (3.41). It is clear that E will become negative when $\beta^2 \sim \beta_H^2 - \mathcal{O}(\ell_s^2/R)$: the Hagedorn temperature will be raised by a factor of order $1/R$.

It might be tempting to speculate that for $R \ll \ell_s$, this tachyon is lifted. In the case of $AdS_5 \times S^5$ compactifications, this regime is dual to a weakly coupled gauge theory. A

similar story was proposed for the tachyon in type 0 theories [47–49]. On the other hand, in those theories, there is evidence that the instability persists in the weakly-coupled large- N gauge theory [50, 51] as a Coleman-Weinberg-type instability. In the case of large- N field theories at finite temperature on S^3 (dual to AdS_5 in global coordinates), the Hagedorn transition is also known to exist at weak coupling [14, 18, 19].

4. The Hagedorn transition in anti-de Sitter spacetime

In this section we will study the Hagedorn transition in AdS spacetime. In flat space, above the Hagedorn temperature, the thermal scalar becomes a tachyon. One expects the high-temperature phase on Euclidean space to correspond to a vev for this scalar. This is analogous to the statement [52, 53] that the Polyakov-Susskind loop $P \text{tr} e^{-\oint A_0 dx^0}$ (or its norm squared [14]) gets an expectation value in the deconfined phase of gauge theories. In string theory, this transition is highly problematic in flat space — as discussed in the introduction and in [4], the canonical ensemble does not strictly exist for interacting strings even below the Hagedorn temperature; while above the Hagedorn temperature, the condensation involves free energies of order the inverse gravitational coupling, and computational control is lost.

However, for string theory in anti-de Sitter spacetimes the AdS curvature provides a “box” in which to study string theory, and provides a dual field theory description for which one expects thermodynamics to be well defined. For example, as we will argue below following refs [54, 21, 10, 23], the analogy between the thermal scalar and the Polyakov-Susskind loop can be made precise in anti-de Sitter spacetimes which are dual to (conformal) gauge theories. (These conformal field theories do not confine, but the large ’t Hooft coupling theories do exhibit charge screening at high temperatures [56], and the theories at weak and strong coupling do see a jump in their free energies from $\mathcal{O}(1)$ to $\mathcal{O}(N^2)$ [55, 54, 14, 18].) This is related to the statement, discussed in [21, 22, 10] that the natural endpoint of the condensation process is the Euclidean black hole.

In anti-de Sitter space at the Hagedorn transition, the wavefunction for the tachyonic thermal scalar is localized near the origin, where the mass is smallest. The condensation of localized tachyonic winding modes has been studied recently [23] with the conclusion that the winding direction pinches off. As we will discuss, in thermal AdS space times this is consistent with the endpoint being the AdS black hole.

4.1 Endpoints of the condensation of the thermal scalar

The following picture of the thermodynamics of string theory on global AdS_5 appears to hold at weak and strong coupling [55, 54, 14, 18, 19]. As the temperature reaches $1/R$, large and small black holes become extrema of the free energy. At a higher temperature also of order $1/R$, the theory undergoes a first-order Hawking-Page phase transition [57] and a “large” black hole with horizon radius $r_s > R$ becomes thermodynamically preferred. The “small” black hole solution has horizon radius less than R ; as one increases the temperature beyond the Hawking-Page transition temperature, its free energy is larger than that of thermal AdS or of the large AdS-Schwarzschild black hole. As argued by [21, 22, 10, 19], at

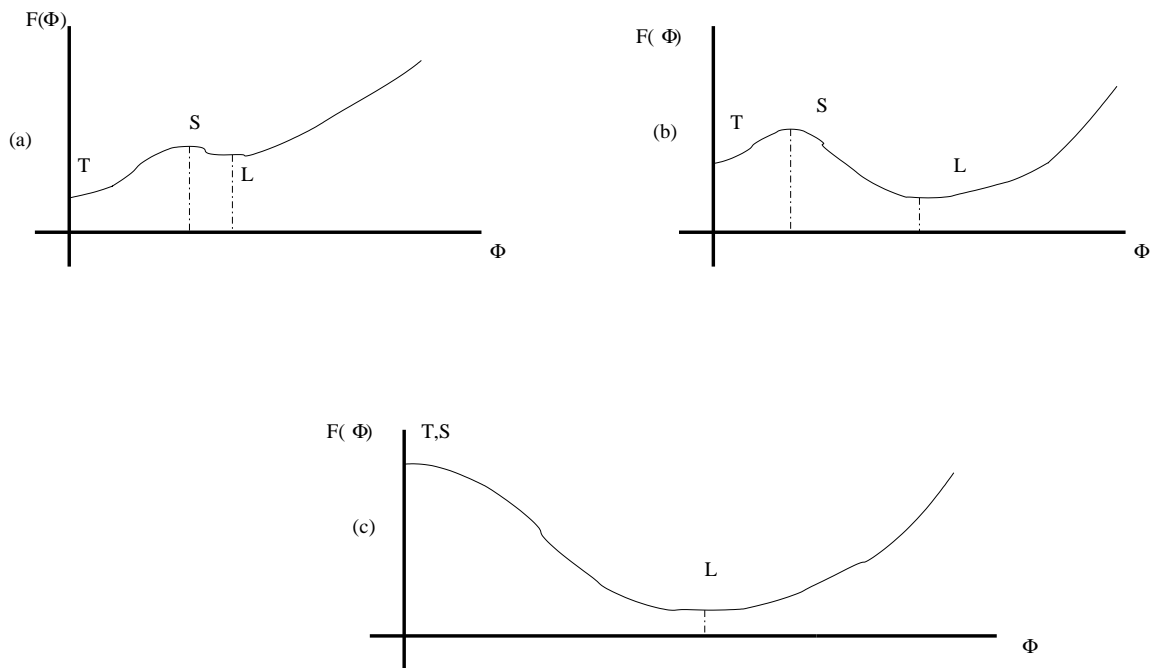


Figure 4: The free energy $F(\Phi)$ as a function of the log of the norm of $W(C)$, as the temperature increases. The labels T,S, and L label the extrema of the free energy corresponding to “thermal AdS”, the “small” black hole, and the “large” black hole, respectively. Figure (a) represents the free energy at a temperature just above that at which the black hole solutions begin to exist. Figure (b) represents a temperature above the Hawking-Page transition temperature, for which the large black hole is thermodynamically stable. Figure (c) represents the Hagedorn transition, at which T and S merge and thermal AdS becomes a local maximum of the free energy.

temperatures of order the Hagedorn temperature, the small black hole and thermal AdS solutions coalesce. At these temperatures the horizon of the small black hole will have curvatures of order the string scale, and should undergo a transition to a long fundamental string [58].

The natural picture of the free energy is shown in figure 4. The order parameter shown is related to the Polyakov-Susskind loop in the gauge theory. More precisely, we want to compute something like the vev of the norm of the “Maldacena-Wilson” loop that includes the adjoint scalars in the $\mathcal{N} = 4$ vector multiplet:

$$W(C) = \text{tr} \exp \left(i \oint dt (\dot{x}^\mu(t) A_\mu - i y_i(t) \Phi^i) \right) . \tag{4.1}$$

This can be computed by studying worldsheets wrapping the thermal circle in Euclidean $AdS_5 \times S^5$ [56, 59, 60, 64]. It is important that we compute the norm [14]: in finite volume, one should integrate over the phase of the $W(C)$, leading to a vanishing expectation value [54, 14]. The statement in the bulk dual at strong coupling is that one should integrate over the value of the NS-NS B-field through the “cigar” parameterized by t, ρ [54].

We would like to relate a nonvanishing vev for the norm of $W(C)$ to a condensate of thermal scalars in the bulk. The basic argument is as follows. The expectation value of

$W(C)$ in the gauge theory is dual to the classical action of a fundamental string worldsheet which wraps the Euclidean time direction in the bulk [54, 56, 59–62].³ In AdS spacetimes, no such classical solution exists. In the presence of a condensate of winding modes, however, the worldsheet can end at the condensate, since a condensate of winding modes spontaneously breaks winding number.

On the other hand, at finite temperature the expectation value of this loop can be calculated in the dual supergravity backgrounds corresponding to extrema of the free energy, without reference to the thermal scalar. The existence of nontrivial solutions arises in the presence of black holes, which change the topology of the spacetime so that a circle along the Euclidean time direction can be contracted to a point at the black hole horizon.

We can show that at strong coupling, $\ln \sqrt{\langle |W|^2 \rangle} = r_s$ for a black hole with radius r_s . One performs the bulk computation of $W(C)$ by computing the action of a string worldsheet which wraps the t, ρ plane at fixed angular coordinate [54, 56, 59–62] and which asymptotes to the curve C at the boundary of AdS.⁴

To compute the norm squared of $W(C)$, we will compute the correlator of two such loops placed on the antipodes of the spatial S^3 in the gauge theory. At large N , the disconnected part of the correlator should dominate, so that

$$\langle W(C)W^\dagger(\tilde{C}) \rangle \equiv \langle |W(C)|^2 \rangle \tag{4.2}$$

where if C is a curve along Euclidean time and localized at a point $x \in S^3$, then \tilde{C} is the curve along Euclidean time localized at the antipode of x in the S^3 .

In general these worldsheets are infinite due to the large- ρ limit, corresponding to UV divergences in the field theory calculation [56, 64]. We will regulate these by subtracting the contribution to $\langle W(C_1)W(C_2) \rangle$ in thermal AdS.⁵ The result is that the regulated correlator vanishes for the thermal AdS contribution, reflecting the fact that the topology of thermal AdS forbids a classical worldsheet contributing to $\langle W(C) \rangle$ even before integrating over the NS-NS B field.

The metric of both AdS space and the AdS black hole can be written as:

$$ds^2 = f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega_3^2 \tag{4.3}$$

For AdS space, $f(r) = 1 + \frac{r^2}{R^2}$. For the AdS_5 black hole with mass M ,

$$f(r) = 1 + \frac{r^2}{R^2} - \frac{8G_N M}{3\pi r^2} = 1 + \frac{r^2}{R^2} - \frac{wM}{r^2} \tag{4.4}$$

³The calculations listed in the references above were done in Poincaré coordinates, and aside from [54] were concerned with Wilson loops or with computing the quark-antiquark potential. A calculation of this potential in global AdS can be found in [63].

⁴In principle our calculation is too quick. For example, we are ignoring $y(t)$ in (4.1). In general, expectation values and correlators of $W(C)$ have a linear divergence unless $\dot{x}^2 = \dot{y}^2$ [64]. Nonetheless, we are interested in the behavior of the string worldsheet deep in the interior of the AdS and AdS-Schwarzschild geometries. Therefore we expect our calculation, with our cutoff, to be qualitatively correct.

⁵Again, see the caveats in the previous footnote

Here $G_N \sim \ell_{p,5}^3$ is the five-dimensional Newton's constant. $\ell_{p,5}$ is the five-dimensional Planck length. Note that $R^5 \ell_{p,5}^3 = \ell_{p,10}^8$, where $\ell_{p,10}$ is the ten-dimensional Plank scale.

The Schwarzschild radius r_h , the location of the horizon in these coordinates, is the largest root of $f = 0$,

$$r_h^2 = \frac{R^2}{2} \left(-1 + \sqrt{1 + \frac{4wM}{R^2}} \right) \tag{4.5}$$

Note that for $\ell_{p,5}^3 M \ll R^2$, $r_h^2 \sim 2\ell_{p,5}^2 M \ll R^2$. The black hole is much smaller than the AdS radius of curvature and is well-approximated by the 5d Schwarzschild black hole in flat space. For $\ell_{p,5}^3 M \gg R^2$, $r_h^2 \sim 2R\sqrt{wM}$ and the black hole is a 5d AdS-Schwarzschild black hole with size much larger than the AdS radius.

The corresponding temperature is set by the requirement that the metric be nonsingular when $f = 0$. This leads to:

$$\beta = \frac{4\pi R^2 r_h}{4r_h^2 + 3R^2} \tag{4.6}$$

Solving for r_h we find two radii for a given temperature:

$$r_h = \frac{\pi R^2}{2\beta} \left(1 \pm \sqrt{1 - \frac{3\beta^2}{\pi^2 R^2}} \right) \tag{4.7}$$

which correspond to the “small” and “large” black holes with radii smaller and larger, respectively, than the AdS radius. Note that r_h represents the minimum of the coordinate r . When $\beta \ll R$, well above the Hawking-Page transition temperature,

$$r_h = \frac{\pi R^2}{2\beta} \quad \text{or} \quad \frac{3\beta}{2\pi} \tag{4.8}$$

for large and small black holes, respectively.

We compute the action of the string wrapping the r, t coordinates by computing the Nambu-Goto action. Let the worldsheet coordinates be τ, σ so that $t = \tau$, $r = \sigma$. Then

$$S = \int_0^\beta d\tau \int_{r_h}^{r_{\max}} d\sigma = \beta(r_{\max} - r_h) \tag{4.9}$$

where $r_{\max} \rightarrow \infty$ is a cutoff on the worldsheet action, representing a UV cutoff in the definition of the $W(C)$ [56]. Similarly, the contribution of thermal AdS to the correlator of two such loops placed on antipodes of the S^3 is just $2\beta r_{\max}$. We can subtract half of this to find that $\Phi \equiv \ln \sqrt{\langle |W|^2 \rangle} = 0, r_{h,small}(\beta), r_{h,large}(\beta)$ for thermal AdS, the small black hole, and the large black hole, respectively. In the high temperature limit $\beta \gg R$,

$$\begin{aligned} \Phi &= 0 && \text{for thermal AdS} \\ &= \frac{3}{2\pi T} && \text{for the “small” black hole at temperature } T \\ &= \pi R^2 T && \text{for the “large” black hole at temperature } T \end{aligned} \tag{4.10}$$

justifying the identification by [14, 18, 19] of the maxima and minima of the free energy diagram seen in figure 4.

As we discussed above, we also expect a non-zero expectation value for Φ if the thermal scalar condenses. A worldsheet stretching from C on the boundary along r, t would extend along the r direction. At a point where it met the condensate, the worldsheet would be able to end, due to the presence of winding modes.

To see how the supergravity calculation of Φ and the thermal scalar picture might be related, consider at the Hagedorn temperature for string theory in AdS space. As noted by [19], at temperatures of order the string scale, the “small black hole” solution has a horizon of order the string scale. This is the Horowitz-Polchinski correspondence point [58], and at higher temperatures the black hole is expected to be better described as a long string. On the other hand, the “thermal AdS” solution corresponds to a gas of strings and supergravity particles at finite temperature: close to the Hagedorn transition, the dominant configurations are single very long strings [7–9]. It is reasonable to conjecture that at this temperature the local maximum of the free energy $F(\Phi)$ corresponding to the “small black hole” merges with the metastable minimum corresponding to thermal AdS space. This would correspond to a tachyonic instability in $F(\Phi)$. At the same time, one expects the thermal scalar to become a quasilocalized tachyon in AdS space. A natural guess for the endpoint of this condensation process is the big black hole [21].

Adams *et. al* [23] have studied quasilocalized tachyons corresponding to winding modes on cylinders with antiperiodic boundary conditions for fermions, whose radius becomes smaller than ℓ_s in some region of the cylinder. They have argued that the condensation of the tachyon leads to the “pinching off” of the cylinder, as shown in figure 5. This is consistent with the well-known RG flow of the 2d Gaussian (or “XY”) model on a circle above the Kosterlitz-Thouless transition [65, 66]. The winding modes of the string around the circle are the vortices of the XY model, and as the vortices condense, the radius of the circle is driven to zero. Furthermore, according to [23, 37] the tachyon potential on the worldsheet suppresses fluctuations of the string in the region where the tachyon has condensed, much like the worldsheet description of D-brane decay in [67].

We therefore expect that below the Hagedorn transition, the free energy curve $F(\Phi)$ shown in figure 4 can be swept out by “turning on” the thermal scalar. In thermal AdS it is an irrelevant operator on the worldsheet, but in the presence of the “small” black hole, the tachyon becomes a relevant operator, and can induce an RG flow either back to thermal AdS or to the “big” black hole. Heuristically, a worldsheet wrapping the thermal circle and extending in the bulk can end on the condensate of winding modes. On the other hand, we know that in black hole backgrounds, the vev $\langle W(C)W(\tilde{C}) \rangle$ is nonzero because the thermal circle pinches off at the horizon. By the discussion in the previous paragraph, we expect that these two mechanisms are somehow dual to each other, and in Schwarzschild coordinates the physics of the black hole horizon is related to the physics of the thermal scalar.⁶

Previous work has also indicated that the horizon in supergravity is related to the tachyon condensate. As Dabholkar [70] has discussed, following earlier insights by Susskind

⁶Horowitz [36] suggests that for extremal black holes, the thermal scalar condenses near the horizon and modifies the geometry, providing a possible reconciliation between the string theory [68] and supergravity [69] calculations of the entropy of an extremal black hole.

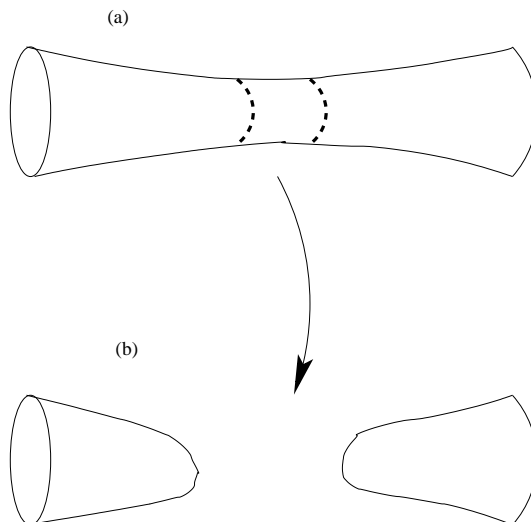


Figure 5: A cylinder with varying radius. The region between the heavy black lines in (a) represents a region for which strings winding around this circle become tachyonic. Figure (b) represents the conjectures endpoint of the condensation of this tachyon.

and Uglum [71], this may help us understand the Bekenstein-Hawking entropy of the black hole. Dabholkar begins with the following observation from [71]. The near-horizon geometry of the Schwarzschild black hole is Rindler space,

$$ds^2 = -\rho^2 dt^2 + d\rho^2 + \dots \tag{4.11}$$

Upon Euclidean continuation, we find that $t \sim t + 2\pi$ is required to avoid a conical deficit angle. This means that Euclidean time has a periodicity equal to the temperature of the black hole. Now the entropy should be $S = \beta^2 \partial_\beta F$, where F is the free energy computed in this background. To vary theta, one must vary the periodicity of t and so introduce a conical deficit angle. Dabholkar computed the free energy as a function of this angle by computing the free energy for the nonsupersymmetric orbifold \mathbb{C}/\mathbb{Z}_N , which has a conical deficit angle $2\pi(1 - \frac{1}{N})$. These orbifolds have twisted sector tachyons, inducing decays from $\mathbb{C}/\mathbb{Z}_N \rightarrow \mathbb{C}/\mathbb{Z}_{N-2}$. The free energy of these orbifolds scale linearly with N . By analytically continuing in N one computes $S = \beta^2 \partial_\beta F$ exactly reproducing the tree-level Bekenstein-Hawking entropy.

The tachyons inducing decay of the orbifold [46] are concentrated at the tip of the cone. In the light of the above description of the “pinching off” of AdS space to form a Euclidean black hole, via condensation of the thermal winding modes, it is natural to identify these localized tachyons as being excitations of the winding mode condensate. The winding modes induced the pinching off of the thermal circle, so it is natural that shifting this condensate should change the geometry of the cones along the lines of [46, 70].

In Lorentzian signature, the thermal scalar has a less direct interpretation. The results of sections 2, 3 indicate that the dominance of the thermal scalar in the partition function near the Hagedorn transition is dual in some sense to the dominance of long strings in the microcanonical ensemble. We speculate that the appearance of closed string tachyons in

computing the black hole entropy is a sign that the entropy is accounted for by long strings near the horizon, after the fashion of [71].

4.2 A conjecture regarding the tachyon potential

In the previous section, following refs. [21, 22, 14, 18, 19], we pointed out that the free energy diagram for finite-temperature strings in anti-de Sitter space could be written as a function of the expectation value of a variant of the Polyakov-Susskind loop, and that this could be related to the bulk expectation value of the thermal scalar. In particular, at the Hagedorn transition, the free energies of the small black hole and of the long string dominating the entropy of thermal AdS merge. The free energy of thermal AdS as a function of the norm of the vev of $W(C)$ becomes a local maximum precisely at the point the thermal scalar in the bulk becomes tachyonic. The natural endpoint of the condensation of the thermal scalar is the large black hole, and there is some evidence that the tachyon accounts for the Bekenstein-Hawking entropy.

We will use this conjecture to estimate the potential energy of the tachyon condensate (assuming that such a potential makes sense), by demanding that it reproduce the free energy of the large black hole. The scaling of this dependence with $R_{\text{ads}}, \ell_{p,10}$, where R_{ads} is the radius of curvature of $AdS_5 \times S^5$, will imply a coupling between the tachyon potential and the spacetime Ricci scalar.

The free energy difference between thermal AdS and the “big” black hole is:

$$I = \beta F \propto -\frac{r_h^3}{\ell_{p,5}^3} \tag{4.12}$$

in the limit $r_h \gg R_{\text{ads}}$. We propose to equate this to the free energy difference due to the thermal scalar potential energy $V(\phi)$. We will assume that inside a region of the size $R^5 r_h^4$ in nine spatial dimensions (we are including the S^5 factor), the tachyon potential V is at its minimum, and outside this region it is at its maximum. The difference δV then scales as

$$\delta V \sim \frac{1}{R_{\text{ads}}^7 \ell_{p,5}^3} \sim \frac{1}{R_{\text{ads}}^2 \ell_{p,10}^8} . \tag{4.13}$$

The dependence on $\ell_{p,10}^8 = g_s^2 \ell_s^8$, with g_s the ten-dimensional string coupling and ℓ_s the string scale, is correct if this potential is generated at tree level in closed string theory. The scaling with g_s^2 of this tree-level free energy is consistent with the results in [4]. The dependence on the AdS radius is puzzling. In particular δV seems to disappear in the flat space limit. In itself this latter point should be related to the fact that the “large” AdS black hole isn’t a solution in flat space. On the other hand, one might have expected a term which scales as $1/g_s^2 \ell_s^{10}$ corresponding to the tachyon potential in flat space.

The scaling (4.13) will be arise if there is a coupling

$$S_{\text{tachyon}} = \int d^{10}x \sqrt{g} V(\phi) R, \tag{4.14}$$

where R is the ten-dimensional Ricci scalar. Couplings of this type for the tachyon of the bosonic string were calculated using sigma model techniques in [72, 73].⁷ The lack of a

⁷Early computations of the tachyon potential can be found in [74 – 77].

term surviving the $R_{\text{ads}} \rightarrow \infty$ limit is still a mystery. We note, however, that Yang and Zwiebach [78] claim that the tachyon potential vanishes both in the perturbative closed string vacuum and after the tachyon has condensed.

We find this conjectured coupling interesting, as similar couplings appear between massless open strings on unstable D-branes and the open string tachyons mediating the decay of these branes. The tree-level effective action for massless and tachyon modes in this system is well described by the Born-Infeld-like Lagrangian:[25, 67, 79–82]

$$S = \int d^d x V(T) \sqrt{\det(\eta_{\mu\nu} - 2\pi\alpha' F_{\mu\nu})} \tag{4.15}$$

Here $V(T)$ vanishes as the tachyon condenses and the branes disappear, indicating that the open string degrees of freedom disappear from the spectrum [25, 24, 26–35].

A similar story arises when one studies the worldsheet theory of open strings in the presence of a tachyon background, discussed in [67, 79]. In that work, the tachyon condensate led to a potential on the boundary of the worldsheet, making it energetically unfavorable for the boundary of the string to live where the open string tachyon condenses. Adams *et al.* [23] and McGreevy and Silverstein [37]. have argued that a similar phenomenon occurs for localized tachyonic winding states in closed string theory.⁸ In particular, the tachyon condensate induces a potential on the worldsheet, suppressing fluctuations of the string into regions with nonzero condensate. Worldsheet correlation functions have support away from this condensate, indicating that closed string amplitudes vanish because the closed strings cease to become dynamical when the tachyon condenses. Indeed, if the condensate of winding tachyons describes the Euclidean black hole, the spacetime ends at the horizon, which we wish to identify as the location of or boundary of the tachyon condensate.

If the proposed action (4.14) is correct, and if $V(T)$ vanishes at the minimum of the tachyon potential, a possible interpretation of this state is that the closed string degrees of freedom disappear, along the lines of the open string scenario. This is consistent with indications based on the worldsheet theory [84, 85, 83, 67].

A serious argument for this gravitational “nothing state” as the endpoint of tachyon condensation will require more than the existence of (4.14). A fuller test of this proposal would be to compute the coupling of the tachyon to the full effective action for the closed string graviton and other massless closed string modes. If this action is, schematically

$$S = \int d^d x \sqrt{g} V(\phi) F(R) \tag{4.16}$$

then we have a better indication that the gravitational dynamics truly decouples at the endpoint $V = 0$ of the condensation process. It would be very interesting to check this coupling from either the worldsheet point of view (perhaps computing multipoint correlators along the lines of [37]) or via string field theory.⁹

⁸A similar proposal has been made for the bosonic string tachyon by Yang and Zwiebach, based on calculations in closed string field theory [78].

⁹Some preliminary calculations for localized tachyons in closed string field theory appear in [86].

5. Conclusions

In the first part of this work, we demonstrated that the dynamics of the thermal scalar for finite-temperature string theory was dual to the dynamics of long strings, and we used this duality to develop a picture of thermally excited strings as random walks describing the configurations of these strings. Following this, we made some observations regarding the endpoint of the condensation of the thermal scalar in AdS spacetimes where string thermodynamics is under some control. In particular, we have argued that the thermal scalar condensate is somehow dual to the AdS-Schwarzschild black hole. It would be interesting to understand this relationship better. The relationship between the thermal scalar and long strings might give a better microscopic picture of the origins of black hole entropy. It would also be interesting to pursue further the suggested decoupling of closed string dynamics when the Hagedorn tachyon condenses.

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A. An identity for the one- and two-string partition functions

Let us study the two-string partition function. Choose a basis of energy eigenstates $|k\rangle$ for a single string such that the Hamiltonian H has energy eigenvalues E_k . The basis elements will include different string modes. The partition function for a single string is

$$Z_{\beta,1} = \sum_k e^{-\beta E_k} \tag{A.1}$$

A basis of states for two bosonic strings with Hamiltonian $H = H_A + H_B$ (where H_A, H_B have identical spectra) is:

$$|k, l\rangle = \frac{1}{\sqrt{2}} (|k\rangle_A |l\rangle_B + |l\rangle_B |k\rangle_A) \quad k \neq l \tag{A.2}$$

$$|k, k\rangle = |k\rangle_1 |k\rangle_2 \tag{A.3}$$

The partition function for two strings is

$$Z_{\beta,2} = \sum_{k>l} e^{-\beta(E_k+E_l)} + \sum_k e^{-2\beta E_k} \quad (\text{A.4})$$

$$= \frac{1}{2} \left(Z_{\beta,1}^2 - \sum_k e^{-2\beta E_k} \right) + \sum_k e^{-2\beta E_k} \quad (\text{A.5})$$

$$= \frac{1}{2} \left(Z_{\beta,1}^2 + \sum_k e^{-2\beta E_k} \right) \quad (\text{A.6})$$

Therefore

$$Z_{\beta,2} - \frac{1}{2} Z_{\beta,1}^2 = \frac{1}{2} Z_{2\beta,1}^2 \quad (\text{A.7})$$

Inspection of (2.2), (2.3) reveals that $Z_{2\beta} = 2Z_{[0,2]}$, proving the last line of (2.6).

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